Dynamic Games with Asymmetric Information: Experience Based Equilibria.

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Based on

• "Experience Based Equilibria", 2012, Chaim Fershtman and Ariel Pakes

• "Methodology for Analyzing Market Dynamics", 2015, Ariel Pakes.

• "Dynamic Procurement Auctions and the Market Impact of Information Sharing" 2018, John Asker, Chaim Fershtman, Jihye Jeon, and Ariel Pakes.

Background: Methodological Developments in IO.

- We have been developing tools that enable us to better analyze market outcomes.

- Common thread: emphasis on incorporating the institutional background needed to make sense of the data used in analyzing the likely causes of historical events, or the likely responses to environmental and policy changes.

- Focus. **Incorporate**
  (i) **heterogeneity** (in plant productivity, products demanded, bidders and/or consumers) and,
  (ii) **equilibrium conditions** when we need to solve for variables that firms could change in response to the environmental change of interest.

We largely relied on earlier work by our game theory colleagues for the analytic frameworks.

- Each agent's actions affect all agents' payoffs, and

- At the "equilibrium" or "rest point"
  (i) agents have correct perceptions, and
  (ii) the system is in some form of "Nash" equilibrium (policies such that no agent has an incentive to deviate).

- Our contribution is the development of an ability to adapt the analysis to the richness of different real world institutions.

The difficulties encountered in incorporating sufficient heterogeneity and/or using equilibrium conditions differed between "static" and "dynamic" models. We were less successful in adapting these models to empirically analyzing dynamic issues.

The initial frameworks by our theory colleagues made assumptions which insurred that the

1. state variables evolve as a Markov process

2. and the equililbrium is some form of Markov Perfection (no agent has an incentive to deviate at any value of the state variables).

E.g. Maskin and Tirole (1988) for theory and Ericson and Pakes (1995) for applied framework. We now consider each of these in turn.

On the Markov Assumption. Except in situations involving active experimentation to learn (where policies are transient), we are likely to stick with the assumption that states evolve as a time homogenous finite order Markov process. Reasons:

- It is a convenient and fits the data well.
- Realism suggests information access and retention conditions limit the memory used.
- We can bound unilateral deviations (Ifrach and Weintraub, 2014), and have conditions which insure those deviations can be made arbitrarily small by letting the length of the kept history grow (White and Scherer, 1994).

On 2: Perfection. The type of rationality built into Markov Perfection is more questionnable; even though it has been useful in the simple models used by our theory and computational colleagues to explore possible outcomes in a structured way, and to generate solutions to selection problems that appear in what are essentially static estimation problems.

I want to start from the premise that the complexity of Markov Perfection not only limits our ability to do dynamic analysis of market outcomes it also

- leads to a question of whether some other notion of equilibria will better approximate agents' behavior.

So the fact that Markov Perfect framework becomes unwieldily when confronted by the complexity of real world institutions, not only limits our ability to do empirical analysis of market dynamics

- it also raises the question of whether some other notion of equilibrium will better approximate agents' behavior.

Question. If we abandon Markov Perfection can we both

- better approximate agents' behavior and,

- enlarge the set of dynamic questions we are able to analyze.

The complexity issue. When we try to incorporate "essential" institutional background we find that the agent is required to:
• Access a large amount of information (all state variables), and
• Either compute or learn an unrealistic number of strategies (one for each information set).

How demanding is this? Consider markets where consumer, as well as producer, choices are dynamic (e.g.'s; durable, experience, or network goods); need the distribution of; current stocks $\times$ household characteristics, production costs, ... In a symmetric information MPE an agent would have to access all state variables, and then either compute a doubly nested fixed point, or learn and retain, policies from each distinct information set.

Obvious Fix: Assume agents only have access to a subset of the state variables.

- Since agents presumably know their own characteristics and these tend to be persistent, we would need to allow for assymetric information: the "perfectness" notion would then lead us to a "Bayesian" Markov Perfect solution.

Is assuming ”Bayesian MP” more realistic? It decreases the information access and retention conditions but increases the burden of computing the policies significantly over the burden of computing in symmetric information MPE models. The additional burden results from the need to compute posteriors, as well as optimal policies; and the requirement that they be consistent with one another.

Could agents learn these policies? I will come back to the issue of what the agents could and could not learn below.

Rest of Talk.

- I am going to introduce a notion of equilibrium that is less demanding than Markov Perfect for both the agents, and the analyst, to use and show how to
  (i) compute the equilibrium and
  (ii) estimate off of equilibrium conditions.

- Consider restrictions that mitigate multiplicity issues.

- Provide a computed example of this equilibrium (dynamic auctions).

I start with strategies that are "rest points" to a dynamical system. Later I will consider institutional change, but only changes where it is reasonable to model responses to the change with a simple reinforcement learning process (I do not consider changes that lead to active experimentation). This makes my job much easier because:

- Strategies at the rest point likely satisfy a Nash condition of some sort; else someone has an incentive to deviate.

- However it still leaves opens the question: What is the form of the Nash Condition?

What Conditions Can We Assume for the Rest Point at States that are Visited Repeatedly?

We expect (and I believe should integrate into our modelling) that

1. Agents perceive that they are doing the best they can at each of these points, and that

2. These perceptions are at least consistent with what they observe.

Note. It might be reasonable to assume more than this: that agents (i) know and/or (ii) explore, properties of outcomes of states not visited repeatedly. I come back to this below.

Formalization of Assumptions.

• Denote the information set of firm i in period t by $J_{i,t}$. $J_{i,t}$ will contain both public (ξ_t) and private ($\omega_{i,t}$) information, so $J_{i,t} = \{\xi_t, \omega_{i,t}\}$.

• Assume $(J_{1,t}, \ldots J_{n_t,t})$ evolves as a finite state Markov process on $\mathcal{J}$ (or can be adequately approximated by one).

• Policies, say $m_{i,t} \in \mathcal{M}$, will be functions of $J_{i,t}$. For simplicity assume $\#\mathcal{M}$ is finite, and that it is a simple capital accumulation game, i.e. $\forall (m_i, m_{-i}) \in \mathcal{M}^n$, & $\forall\, \omega \in \Omega$

$$P_\omega(\cdot|m_i, m_{-i}, \omega) = P_\omega(\cdot|m_i, \omega),$$

(relaxed when we consider auctions below). The public information, ξ, is used to predict competitor behavior and common demand and cost conditions (these evolve as an exogenous Markov process).

• A "state" of the system, is

$$s_t = \{J_{1,t}, \ldots, J_{n_t,t}\} \in \mathcal{S},$$

$\#\mathcal{S}$ is finite. $\Rightarrow$ any set of policies will insure that s_t will wander into a recurrent subset of $\mathcal{S}$, say $\mathcal{R} \subset \mathcal{S}$, in finite time, and after that $s_{t+\tau} \in \mathcal{R}$ w.p.1 forever. Note that the agents does not keep track of all of s_t, only $J_{i,t}$.

• Let the agent's perception of the expected discounted value of current and future net cash flow were it to chose m at state J_i, be

$$W(m|J_i), \ \ \forall m \in \mathcal{M} \ \ \& \ \ \forall J_i \in \mathcal{J},$$

• and of expected profits be

$$\pi^E(m|J_i).$$

Our assumptions imply:

- Each agent choses an action which maximizes its perception of its expected discounted value, and

- For those states that are visited repeatedly (are in $\mathcal{R}$) these perceptions are consistent with observed outcomes.

Formally

A. $W(m^*|J_i) \geq W(m|J_i), \;\; \forall m \in \mathcal{M} \;\; \& \;\; \forall J_i \in \mathcal{J},$

B. &, $\forall J_i$ which is a component of an $s \in \mathcal{R}$

$$W(m(J_i)|J_i) = \pi^E(m|J_i) + \beta \sum_{J_i^{'}} W(m^*(J_i^{'})|J_i^{'}) p^e(J_i^{'}|J_i),$$

where, if $p^e(\cdot)$ provides the empirical probability (the fraction of periods the event occurs)

$$\pi^E(m|J_i) \equiv \sum_{J_{-i}} E[\pi(\cdot)|J_i, J_{-i}] p^e(J_{-i}|J_i),$$

and

$$\left\{ p^e(J_{-i}|J_i) \equiv \frac{p^e(J_{-i}, J_i)}{p^e(J_i)} \right\}_{J_{-i}, J_i},$$

while

$$\left\{ p^e(J_i^{'}|J_i) \equiv \frac{p^e(J_i^{'}, J_i)}{p^e(J_i)} \right\}_{J_i^{'}, J_i}. \;\; \spadesuit$$

“Experience Based Equilibrium”

These are the conditions of a (restricted) EBE (Fershtman and Pakes, 2012; for related earlier work see Fudenberg and Levine, 1993 on self confirming equilibria). Bayesian Perfect satisfy them, but so do weaker notions. We now turn to its :
(i) computational and estimation properties,
(ii) overcoming multiplicity issues,
(iii) and then to an example.

Computational Algorithm. Asynchronous “Reinforcement learning” algorithm (Pakes and McGuire, 2001). Can be viewed as a learning process. Makes it a candidate to:
(i) analyze (small) perturbations to the environment, as well as
(ii) to compute equilibrium. In this context it formally circumvents the traditional sources of the curse of dimensionality; but there is still lots of room for improvement.

Iterations defined by
• A location, say $L^k = (J_1^k, \ldots J_{n(k)}^k) \in \mathcal{S}$: is the information sets of agents active.
• Objects in memory (i.e. M^k):
(i) perceived evaluations, W^k,
(ii) No. of visits to each point, h^k.

Must update (L^k, W^k, h^k)**.** Computational burden determined by; memory constraint, and compute time. I use a simple (not neccesarily) optimal structure to memory.

Update Location.

• Calculate "greedy" policies for each agent

$$m_{i,k}^* = arg \max_{m \in \mathcal{M}} W^k(m|J_{i,k})$$

• Take random draws on outcomes conditional on $m_{i,k}^*$: i.e. if we invest in "payoff relevant"

$\omega_{i,k} \in J_{i,k}$, draw $\omega_{i,k+1}$ conditional on $(\omega_{i,k}, m^*_{i,k})$.
• Use outcomes to update $L^k \rightarrow L^{k+1}$.

Update W^k.

• "Learning" interpretation: Assume agent observes $b(m_{-i})$ and knows the primitives; $\pi_i(\cdot), p(\omega_{i,t+1}|\omega_{i,t}, m_{i,t})$.

• Its ex poste perception of what its value would have been had it chosen m is

$$V^{k+1}(J_{i,k}, m) =$$

$$\pi(\omega_{i,k}, m, b(m_{-i,k}), d_k) + \max_{\tilde{m} \in M} \beta W^k(\tilde{m}|J_{i,k+1}(m)),$$

where $J_i^{k+1}(m)$ is what the $k+1$ information would have been given m and *competitors actual play*.

Treat $V^{k+1}(J_{i,k})$ as a random draw from the possible realizations of $W(m|J_{i,k})$, and update W^k as in stochastic integration (Robbins and Monroe,1956)

$$W^{k+1}(m|J_{i,k}) =$$

$$\frac{1}{h^k(J_{i,k})}V^{k+1}(J_{i,k}, m) + \frac{(h^k(J_{i,k}) - 1)}{h^k(J_{i,k})}W^k(m|J_{i,k}),$$

or

$$W^{k+1}(m|J_{i,k}) - W^k(m|J_{i,k}) =$$

$$\frac{1}{h^k(J_{i,k})}[V^{k+1}(J_{i,k}, m) - W^k(m|J_{i,k})].$$

(other weights are more efficient, it would be good to know how to aggregate states ....)

Notes.

- If we have equilibrium valuations we tend to stay their, i.e. if * designates equilibrium

$$E[V^*(J_i, m^*)|W^*] = W^*(m^*|J_i).$$

• To learn equilibrium values we need to visit points repeatedly; only likely for states in $\mathcal{R}$.

• Agents (not only the analyst) could use the algorithm to find equilibrium policies or adjust to perturbations in the environment.

• Algorithm has no curse of dimensionality.
(i) Computing continuation values: integration is replaced by averaging two numbers.
(ii) States: algorithm eventually wanders into $\mathcal{R}$ and stays their, and $\#\mathcal{R} \leq \#\mathcal{J}$.

• The stochastic approximation literature for single agent problems often augments this with functional form approximations ("TD learning"; Sutton and Barto,1998). The computational burden can still be quite large, so a way of decreasing it would be useful.

Convergence and Testing.

• Testing. The algorithm does not necessarily converge, but a test for convergence exists and does not involve a curse of dimensionality (Fershtman and Pakes, 2012).

• The test is based on simulation. It produces a consistent estimate of an $L^2(P(\mathcal{R}))$ norm of the percentage bias in the implied estimates of $V(m, J_i)$; where $P(\mathcal{R})$ is the invariant measure on the recurrent class.

• **Basis.** Simulate sample paths and $\forall(m, J_i)$ store mean $(\tilde{W}(m|J_i))$ and variance $(\tilde{V}(\tilde{W}(m|J_i))$ of EDV of playing m at J_i. $(\tilde{W}(m|J_i) - W(m|J_i))^2$ is the MSE of $\tilde{W}(m|J_i)$ as an estimate of $W(m|J_i)$.

$$\%Bias^2(m|J_i) = \frac{(\tilde{W}(m|J_i) - W(m|J_i))^2}{W(m|J_i)^2} - \frac{\tilde{V}(\tilde{W}(m|J_i)}{W(m|J_i)^2}.$$

Details. Any fixed W, say $\tilde{W}$, generates policies which define a finite state Markov process for $\{s_t\}$. Gather the transition probabilies into the Markov matrix, $Q(s', s|\tilde{W})$.

To test if the process satisfies our equilbirium conditions need:
(i) a candidate for $\mathcal{R}$, and checks for
(ii) optimality of policies and
(iii) consistency of W.

Candidate for $\mathcal{R}(\tilde{W})$. Start at any s^0 and use $Q(\cdot,\cdot|\tilde{W})$ to simulate a sample path $\{s^j\}_{j=1}^{J_1+J_2}$. Let $\mathcal{R}(J_1, J_2, \cdot)$ be the set of states visited at least once between $j = J_1$ and $j = J_2$.

$$(J_1, J_2) \rightarrow (\infty, \infty), \;\; \& \;\; J_2 - J_1 \;\; \rightarrow \infty$$

$$\Rightarrow \;\; \mathcal{R}(J_1, J_2, \cdot) \rightarrow \tilde{\mathcal{R}},$$

a recurrent class of $Q(\cdot,\cdot|\tilde{W})$ (C1 satisfied).

C2 (optimality of policies). Satisfied by construction, since we use the policies generated by $\tilde{W}$ to form $Q(\cdot,\cdot|\tilde{W})$.

C3 (consistency of $\tilde{W}$ with outcomes). Does

$$\tilde{W}(m^*|J_i) = \pi^E(J_i) + \beta \sum_{J_i^{'}} \tilde{W}(m^*(J_i^{'})|J_i^{'}) p^e(J_i^{'}|J_i)$$

$(\forall J_i \in s \in \mathcal{R}.)$?

Direct summation. Computationally burdensome; indeed brings the curse of dimensionality back in.

Alternative. Check for consistency of simulated sample paths with evaluations.

• Start at $s_0 \in \mathcal{R}$ and forward simulate. At each J_i compute perceived values (our $V^{k+1}(\cdot)$), keep track of the average and the sample variance of those simulated perceived values, say

$$\Big(\widehat{\mu}(\tilde{W}(m^*(J_i)|J_i)), \widehat{\sigma}^2(\tilde{W}(m^*(J_i)|J_i))\Big).$$

• Let $E(\cdot)$ take expectations over the simulated random draws (where draws will be indexed by a tilde), let l index locations, and note that we can compute $\mathcal{T}_l$, where

$$\mathcal{T}_l \equiv E\Big(\frac{\widehat{\mu}(\tilde{W}_l) - \tilde{W}_l}{\tilde{W}_l}\Big)^2$$

$$= E\Big(\frac{\widehat{\mu}(\tilde{W}_l) - E[\widehat{\mu}(\tilde{W}_l)]}{\tilde{W}_l}\Big)^2 + \Big(\frac{E[\widehat{\mu}(\tilde{W}_l)] - \tilde{W}_l}{\tilde{W}_l}\Big)^2.$$

$$= \%Var(\widehat{\mu}(\tilde{W}_l)) \ + \ \%Bias^2(\widehat{\mu}(\tilde{W}_l)).$$

• $\mathcal{T}_l$ is observed, as is f_l, the fraction of visits to l. As the number of simulation draws grows

$$\sum_l f_l\Big(\frac{\widehat{\sigma}^2(\tilde{W}_l)}{\tilde{W}_l^2}\Big) - \sum_l f_l\Big(\frac{\widehat{\mu}(\tilde{W}_l) - E[\widehat{\mu}(\tilde{W}_l)]}{\tilde{W}_l}\Big)^2 \rightarrow_{a.s.} 0,$$

$\Rightarrow$

$$\sum_l f_l \mathcal{T}_l - \sum_l f_l \Big(\frac{\hat{\sigma}^2(\tilde{W}_l)}{\tilde{W}_l^2}\Big) \rightarrow_{a.s.} \sum f_l \Big(\frac{E[\hat{\mu}(\tilde{W}_l)] - \tilde{W}_l}{\tilde{W}_l}\Big)^2,$$

an $L^2(\mathcal{P}_\mathcal{R})$ norm in the percentage bias ($\mathcal{P}_\mathcal{R}$ is the invariant measure associated with $(\mathcal{R}, \tilde{W})$).

Estimation.

• Need a candidate for J_i. Either:
(i) empirically investigate determinants of controls (determinants of controls), and/or
(ii) ask actual participants.

• Does not require nested fixed point algorithm. Use estimation advances designed for MP equilibria (POB or BBL), or a perturbation (or "Euler" like) condition (below).

Euler-Like Condition.

• With assymetric information the equilibrium condition

$$W(m^*|J_i) \geq W(m|J_i)$$

is an inequality which can generate (set) estimators of parameters.

• J_i contains both public and private information. Let J^1 have the same public, but differnt private, information then J^2. If a firm is at J^1 it knows it could have played $m^*(J^2)$ and its competitors would respond by playing *on the equilbrium path* from J^2.

• If $m^*(J^2)$ results in outcomes in $\mathcal{R}$, we can simulate a sample path from J^2 using only observed equilibrium play. The Markov property insures it would intersect the sample path from the DGP at a random stopping time with probability one and from that time forward the two paths would generate the same profits.

• The conditional (on J_i) expectation of the difference in discounted profits between the simulated and actual path from the period of the deviation to the random stopping time, should, when evaluated at the true parameter vector, be positive. This yields moment inequalities for estimation as in Pakes, Porter, Ho and Ishii (2015), Pakes, (2010).

Multiplicity.

• $\mathcal{R}$ contains both "interior" and "boundary" points. Points at which there are feasible strategies which can lead outside of $\mathcal{R}$ are boundary points. Interior points are points that can only transit to other points in $\mathcal{R}$ no matter which (feasible) policy is chosen.

• Our conditions only insure that perceptions of outcomes are consistent with the results from actual play at interior points. Perceptions of outcomes for some feasible (but inoptimal) policy at boundary points are not tied down by actual outcomes.

• "MPBE" are a special case of (restricted) EBE and they have multiplicity. Here differing perceptions at boundary points can support a (possibly much) wider range of equilibria.

Narrowing the Set of Equilibria.

• In any empirical appliction the data will rule out equilibria. m^* is observable, at least for states in $\mathcal{R}$, and this implies inequalities on $W(m|\cdot)$. With enough data $W(m^*|\cdot)$ will also be observable up to a mean zero error.

• Use external information to constrain perceptions of the value of outcomes outside of $\mathcal{R}$. If available use it.

• Allow firms to experiment with $m_i \neq m_i^*$ at boundary points (as in Asker, Fershtman, Jihye, and Pakes, 2014). Leads to a stronger notion of, and test for, equilibrium. We insure that perceptions are consistent with the results from **actual play** for each **feasible** action at boundary points (and hence on $\mathcal{R}$).

Boundary Consistency.

Let $B(J_i|\mathcal{W})$ be the set of actions at $J_i \in s \in \mathcal{R}$ which could generate outcomes which are not in the recurrent class (so J_i is a boundary point) and $B(\mathcal{W}) = \cup_{J_i \in \mathcal{R}} B(J_i|\mathcal{W})$. Then the extra condition needed to insure "Boundary Consistency" is:

Extra Condition. Let τ index future periods, then $\forall (m, J_i) \in B(\mathcal{W})$

$$W(m^*|J_i) \geq$$

$$E\Big[\sum_{\tau=0}^{\infty} \delta^\tau \pi(m(J_{i,\tau}), m(J_{-i,\tau}))|J_i = J_{i,0}, \mathcal{W}\Big],$$

where $E[\cdot|J_i, \mathcal{W}]$ takes expectations over future states starting at J_i using the policies generated by $\mathcal{W}$. ♠

Testing for Boundary Consistency.

Fix $(m, J_i) \in B(i)$. Simulate independent sample paths from it with initial J_{-i} drawn from the empirical distribution of $p^e(J_{-i}|J_i)$. Calculate mean, $\widehat{W}(m|J_i)$, and the variance, $\widehat{V}(\widehat{W}(m|J_i))$, of simulated sample path for each $(m, J_i) \in B(i)$.

Basis of Test. Average

$$\frac{\Big((W(m^*|J_i) - \widehat{W}(m|J_i))_-\Big)^2}{\widehat{V}(\widehat{W}(m|J_i))}$$

over $(m, J_i) \in B(i)$ and then a weighted average of these over boundary points. This is an Inequalities based test and one needs to simulate the test statistic's critical values.

Each path which we simulate either will or will not re-enter $\mathcal{R}$. Provided prior test is satisfied we have the correct expectation of the future value from any $(J_i, J_{-i}) = s \in \mathcal{R}$.

Let; r index simulation samples,
γ_r index the periods simulated for sample r,
γ_r^* be the first period when $s_{\gamma_r} \in \mathcal{R}$ (or some sufficiently large number if it does not enter),
$\{s_{\gamma_r}\}_{\gamma=1}^{\gamma_r^*}$ be the sequence of states simulated for sample path r.
Then an unbiased estimate of the actual value of the feasible play is

$$\widehat{W}_r(m|J_i) \equiv$$

$$\sum_{\gamma_r=1}^{\gamma_r^*-1} \delta^{\gamma_r} \pi(m(J_{i,\gamma_r}), m^*(J_{-i,\gamma_r})) + \delta^{\gamma_r^*} W(m^*|J_{i,\gamma_r^*}).$$

If there are R simulated paths, let $\overline{W}^R(m|J_i)$ be their average, and $Var[\overline{W}^R(m|J_i)]$ be the standard estimate of the variance of this average.

Let $B(J_i) = \{m : (m, J_i) \in B\}$ and $\#B(J_i)$ be the number of elements in $B(J_i)$. So

$$T(J_i) = \frac{1}{\#B(J_i)} \sum_{m \in B(J_i)} \left(\frac{[\overline{W}^R(m|J_i) - W(m^*|J_i)]_+}{W(m^*(J_i))}\right),$$

is a measure of the deviation of the boundary point from boundary consistency.

Let $\mathcal{J}_B = \{J_i : (b, J_i) \in B$ for at least one $b\}$, $h(J_i)$ be the number of times times the point J_i was visited in the test run, and

$$q(J_i) = \frac{\{J_i \in B\} h(J_i)}{\sum_{J_i \in B} h(J_i)}.$$

Then our test statistic is

$$T(B) = \sum_{J_i \in \mathcal{J}_B} q(J_i) T(J_i).$$

We have to simulate its distribution under the null that $W(m|J_i) = W(m^*|J_i)$ for each $(m, J_i) \in B$ (this insures the size of the test), and check whether the 95^th percentile of the simulated distribution is larger then $T(B)$. We accept

$$H_0 : \text{Boundary Consistency}$$

if and only if it is not.

Eg.: Dynamic Procurement Auctions: The Impacts of Information Sharing. J. Asker, C. Fershtman, J. Jeon, A. Pakes.

Dynamic auctions are sequential auctions in which the state of the bidders, and therefore their evaluation of the good that is auctioned, change endogenously depending on the history of auction.

The value of winning an auction to produce aircraft or ships depends on the backlog or the order book of the firm, and the value of winning a highway repair project or a timber auction depends on whether the inputs currently under the control of the firm are already fully committed for the following period.

Structure of game.

- There is an auction for the right to harvest timber on a parcel of land in each period.

- Firms enter the period with a stock of lumber $\omega_{i,t}$. They harvest, process, and sell at a fixed price of one on the world market in each period. The harvest/processing outcome is stochastic.

- Firms decide whether to pay a fee (F) and enter the auction. Simultaneously those who do enter submit a bid, $b \in \{b_1, \ldots \bar{b}\} = \mathcal{B} \subset \mathcal{Z}_+$.

- If there is information exchange it occurs between the time the bids are submitted, and the outcome of the auction is announced.

When information is shared the reported information is truthful.

- The winner discovers the amount of timber on the plot $[(\theta + \eta);\ \eta \sim F_\eta(\cdot)]$, and each firm gets a random draw on harvest/processing $[(e + \epsilon);\ \epsilon \sim F_\epsilon(\cdot)]$.

- If $\{i_w(J_i, J_{-i})\}$ is one when the firm wins the auction and zero elsewhere

$$\pi(J_i, J_{-i}, \epsilon_i, \eta_i) =$$

$$min\{\omega_i + \{i_w(\cdot)\}(\theta + \eta), e + \epsilon_i\} - \{i_w\}b_i - g(J_i)F.$$

Information Sets.

- Basic question: what are the implications of different information structures in dynamic auctions and do those implications depend on the extent to which we discount the future.

- Compare institutions which generate

  - revelation in each period,

  - revelation every $T > 1$ periods, and

  - every T periods firms chose whether to reveal in each of the next T periods. They both have to want to reveal before any of them reveals. The decision is made just after information revelation.

• Information sets. Let τ_t be the time since last iteration, $i_w(t)$ provide the identity of the winning bidder $b_w(t)$ its bid, p_t be the participation decisions, and $\omega_t = (\omega_{i,t}, \omega_{-i,t})$. Then $J_{i,t} = (\omega_{i,t}, \xi_t)$, and if there is no information revelation

$$\xi_t = \{p_t, i_w(t), b_w(t), \tau_t\} \cup \xi_{t-1},$$

while if there is information revelation

$$\xi_t = \{\omega_{t-1}, i_w(t), b_w(t), \tau_t = 1\}.$$

• Note: this is not a capital accumulation game. I.e. one agent's choice of control will affect the evolution of the other firm's state. This complicates both the computation and the economics; an agent can refrain from bidding today in order to let its competitors' accumulate today so that it will be less aggressive in the future.

Value function

$$V(J_i) = max\{W(0|J_i), \max_{b \in \mathcal{B}} W(b|J_i)\}$$

Let

$$\pi^E(J_i) = \sum_{J_{-i},\eta,\epsilon_i} \pi(J_i, J_{-i}, \epsilon_i, \eta) p(J_{-i}|J_i) p(\epsilon_i) p(\eta).$$

Then if $b_i > 0$ the firms participate and

$$W(b \neq 0|J_i) = \pi^E(J_i) +$$

$$\beta p^w(b|J_i) \sum_{\epsilon_i,\eta,\xi'} V(\omega'(\omega_i, \eta, \epsilon_i), \xi') p(\xi'|J_i, b, i = i_w) p(\eta, \epsilon) +$$

$$\beta(1 - p^w(b|J_i)) \sum_{\epsilon_i,\xi'} V(\omega'(\omega_i, \epsilon_i), \xi') p(\xi'|J_i, b, i \neq i_w) p(\epsilon).$$

If $b_i = 0$ (the firm does not participate)

$$W(0|J_i) = \pi^E(J_i) + \beta \sum_{\epsilon_i,\xi'} V(\omega'(\omega_i, \epsilon_i), \xi') p(\xi'|J_i) p(\epsilon).$$

Parameter Values

Parameters:		B	IE	VIE
Periods between ω revelation	T	4	1	{1,4}
Common Parameters:				
Distribution of fixed cost of participation	F_i		U[0,1]	
Discount factor	β		0.9	
Mean timber in a lot	θ		3.5	
Disturbance around θ	η		{-0.5,0.5}	
Probability on η realizations			{0.5,0.5}	
Mean harvest capacity	e		2	
Disturbance around e	ϵ		{-1,0,1}	
Probability on ϵ realizations			{0.33,0.33,0.33}	
Bidding grid			{0.5,1,1.5,2}	
Number of firms/bidders			2	
Retail price of a unit of timber			1	

Computational Details.

Size of recurrent class:

B	IE	VIE
325,843	2,081	328,692

Number of all states visited during computation:

B	IE	VIE
7,495,307	2,724	7,908,122

Computation times per 5 million iterations (in hours):

B	IE	VIE
1:38	1:06	1:56

Computation times for testing for a REBE (5 million iterations, in hours):

B	IE	VIE
1:43	1:09	2:00

Computation times for testing for boundary consistency (100,000 iterations

B	IE	VIE
3:03	0:16	75:41

Notes: Computation was conducted in MATLAB version R2013a using (a Dell Precision T3610 desktop with) a 3.7 GHz Intel Xeon processor and 16GB RAM on Windows 7 Professional.

Six rounds of computation were required for B to pass the REBE test, eight for VIE and one for IE. We estimated models with several other parameter values. All that past the REBE test but one were boundary consistent, but we started with very high initial conditions.

Summary Statistiics

	B	IE	VIE	SP
Avg. bid	1.09	0.94	1.04	-
Avg. b_w (revenue for the auctioneer)	1.11	0.98	1.07	-
Avg. b_w when ≥ 1 firm	1.16	0.98	1.12	-
Avg. b_w with 1 firm	1.06	0.67	0.99	-
Avg. b_w with 2 firms	1.23	1.16	1.20	-
Avg. # of participants	1.52	1.63	1.52	1
Avg. # of participants, with ≥ 1 firm	1.59	1.63	1.59	1
Avg. participation rate	0.76	0.81	0.76	0.50
% of periods with no participation	4.39	0.15	3.85	0.004
Avg. total revenue	3.35	3.49	3.37	3.50
Avg. profit	0.81	0.87	0.84	-
% of periods; lowest omega wins	66.37	60.80	65.32	85.96
Average total social surplus	2.73	2.72	2.74	3.10

Procurement Revenue = winning bid.
$\pi_i(\cdot) = \min\{\omega_i + \{i_w\}(\theta + \eta), e + \epsilon_i\} - \{i_w\}b_i - g(J_i)F.$
Revenue $\min\{\omega_i + \{i_w\}(\theta + \eta), e + \epsilon_i\}$.
Social surplus $\sum_i [\pi_i(\cdot) - g(J_i)F]$.

- IE has lower bids but more participation. We would not expect that combination in a static auction as we expect less aggressive bidding to occur with less participation.

- VIE is very close to B, indicating that when there is a choice as to whether to exchange information most of the time they do not exchange information. On the other hand the average profit and hence the average value is higher in the setting where information is exchanged.

States and Profits.

(ω_i, ω_{-i})	Prob. Dist. (%) B	IE	SP	Profit B	IE
$(\leq 4, \leq 4)$	65.51	32.59	90.12	0.68	0.52
$(\leq 4, 5-7)$	12.61	19.09	4.52	0.57	0.58
$(\leq 4, \geq 8)$	4.05	10.55	0.28	0.60	0.59
$(5-7, \leq 4)$	12.61	19.09	4.52	1.51	1.26
$(5-7, 5-7)$	0.88	5.72	0.22	1.49	1.46
$(5-7, \geq 8)$	0.14	1.12	0.02	1.49	1.13
$(\geq 8, \leq 4)$	4.05	10.55	0.28	1.62	1.58
$(\geq 8, 5-7)$	0.14	1.12	0.02	1.66	1.87
$(\geq 8, \geq 8)$	0.01	0.17	0.00	1.72	1.56

Notes: This table shows the probability of intervals of ω-tuples for B, IE and SP. Here the per-period profit is a probability weighted average, over the states underlying each ω-tuple.

- B has higher profits in just about every state, yet IE has higher value.
- The reason is that IE spends disproportionate time in states where stocks are higher, bidding is less aggressive and both equilibria have more value.
- The control here is bids; so to understand how the additional info on the competitor enables IE to do this we look at bids.

Differences in Policies.

(ω_i, ω_{-i})	Bids							
	B				IE			
	0	0.5	1	1.5/2	0	0.5	1	1.5/ 2
$(\leq 4, \leq 4)$	0.22	0.13	0.27	0.38	0.07	0.13	**0.28**	**0.53**
$(\leq 4, 5-7)$	0.11	0.32	0.45	0.13	0.02	**0.53**	0.37	0.08
$(\leq 4, \geq 8)$	0.08	0.58	0.29	0.06	0.00	**0.88**	0.12	0.00
$(5-7, \leq 4)$	0.43	0.18	0.34	0.05	0.33	0.10	**0.52**	**0.05**
$(5-7, 5-7)$	0.37	0.50	0.09	0.03	**0.40**	**0.59**	0.01	0.00
$(5-7, \geq 8)$	0.39	0.53	0.06	0.02	0.11	**0.89**	0.00	0.00
$(\geq 8, \leq 4)$	0.51	0.25	0.22	0.02	**0.60**	0.14	**0.26**	0.00
$(\geq 8, 5-7)$	0.53	0.39	0.06	0.01	**0.84**	0.16	0.00	0.00
$(\geq 8, \geq 8)$	0.61	0.36	0.03	0.00	0.47	**0.53**	0.00	0.00

Items in boldface are probabilities that are greater in IE than in B

Note:

- IE bids more intensely at low states, and in middle states when the competitor is lower, but less intensely at middle states when both are middle, and high states.

- This enable more time spent at higher states and at high states the bidder either stays out or bids .5, so these are profitable states. I.e. the extra information allows them to coordinate better at high states, and this provides the incentives to bid so as to get their.

Static Incentives, $\beta = 0$

	$\beta = 0.9$		$\beta = 0$	
	B	IE	B	IE
Avg. bid	1.09	0.94	0.61	0.59
Avg. b_w (revenue for the auctioneer)	1.11	0.98	0.54	0.53
Avg. b_w with $\geq$ 1 firm participating	1.16	0.98	0.62	0.60
Avg. b_w with 1 firm participating	1.06	0.67	0.55	0.53
Avg. b_w 2 firms participating	1.23	1.16	0.82	0.82
Avg. # of participants	1.52	1.63	1.10	1.10
Avg. # of participants with $\geq$ 1 firm	1.59	1.63	1.25	1.25
Avg. participation rate	0.76	0.81	0.55	0.55
% of periods with no participation	4.39	0.15	11.98	11.65
Avg. total revenue	3.35	3.49	3.08	3.09
Avg. profit	0.81	0.87	1.03	1.04
% of periods; lowest ω wins conditional on $\geq$ 1 firm participating	66.37	60.80	96.24	96.15
Average total social surplus	2.73	2.72	2.60	2.61

Notes

- History still matters here, as the information gathered from it is still a signal on competitor's ω.

- Very little difference between B and IE when firms do not care about the future.

- The advantage of extra information on a competitors' likely bid is that it enables better coordination at high states, but to get to them we need to bid more aggressively at middle states, and without a future there is no incentive to do that.

Voluntary Information Exchange.

	(%)	$\Pr(\cup_i \chi_i \geq 1)$	$\Pr(\Pi_i \chi_i = 1$	Profit	
(ω_i, ω_{-i})	VIE	VIE		B	IE
$(\leq 4, \leq 4)$	62.98	24.75	4.76	0.68	0.52
$(\leq 4, 5-7)$	13.17	24.57	4.47	0.57	0.58
$(\leq 4, \geq 8)$	4.58	28.06	6.09	0.60	0.59
$(5-7, \leq 4)$	13.17	21.38	4.47	1.51	1.26
$(5-7, 5-7)$	1.13	18.94	4.59	1.49	1.46
$(5-7, \geq 8)$	0.19	24.38	9.73	1.49	1.13
$(\geq 8, \leq 4)$	4.58	23.39	6.09	1.62	1.58
$(\geq 8, 5-7)$	0.19	24.60	9.73	1.66	1.87
$(\geq 8, \geq 8)$	0.02	38.14	20.34	1.72	1.56

$\chi_i \in \{0,1\}$, $\chi_i = 1$ indicates that firm i chose to reveal, so $\cup_i \chi_i \geq 1$ indicates that at least one firm chose to reveal and $\Pi_i \chi_i = 1$ indicates both firms chose to reveal. Only periods in which firms decide on information sharing (or periods with $\tau = 0$) are used in the calculation.

Notes

- Firms only chose to share info in 5% of the possible states (thought one of the two shares 24% of time).
- Recall that value in IE is higher, so why only 5%?
- The propensity to share info. is only large at high ω states. However we start in B which is predominantly low ω states. In those states profits are higher in B and hence to progress to IE we would have to give up intermediary profits with no guarantee that we will stay at IE four periods hence. Lack of an ability to commit generates this.